

THE CLASSICAL DYNAMIC SYMMETRY FOR THE $U(1)$ -KEPLER PROBLEMS

SOFIANE BOUARROUDJ AND GUOWU MENG

ABSTRACT. For the Jordan algebra of hermitian matrices of order $n \geq 2$, we let X be its submanifold consisting of rank-one semi-positive definite elements. The composition of the cotangent bundle map $\pi_X: T^*X \rightarrow X$ with the canonical map $X \rightarrow \mathbb{C}P^{n-1}$ (i.e., the map that sends a hermitian matrix to its column space), pulls back the Kähler form of the Fubini-Study metric on $\mathbb{C}P^{n-1}$ to a real closed differential two-form ω_K on T^*X . Let ω_X be the canonical symplectic form on T^*X and μ be a real number. A standard fact says that $\omega_\mu := \omega_X + 2\mu\omega_K$ turns T^*X into a symplectic manifold, hence a Poisson manifold with Poisson bracket $\{, \}_\mu$.

In this article we exhibit a Poisson realization of the simple real Lie algebra $\mathfrak{su}(n, n)$ on the Poisson manifold $(T^*X, \{, \}_\mu)$, i.e., a Lie algebra homomorphism from $\mathfrak{su}(n, n)$ to $(C^\infty(T^*X, \mathbb{R}), \{, \}_\mu)$. Consequently one obtains the Laplace-Runge-Lenz vector for the classical $U(1)$ -Kepler problem with level n and magnetic charge μ . Since the McIntosh-Cisneros-Zwanziger-Kepler problems (MICZ-Kepler Problems) are the $U(1)$ -Kepler problems with level 2, the work presented here is a direct generalization of the work by A. Barut and G. Bornzin [*J. Math. Phys.* **12** (1971), 841-843] on the classical dynamic symmetry for the MICZ-Kepler problems.

Keywords. Kepler problem, Jordan algebra, dynamic symmetry, Laplace-Runge-Lenz vector.

1. INTRODUCTION

Let \mathfrak{g} be a real Lie algebra. A **Poisson realization** of \mathfrak{g} on a Poisson manifold M is a Lie algebra homomorphism from \mathfrak{g} to $(C^\infty(M, \mathbb{R}), \{, \})$. It has been known for more than 40 years that $\mathfrak{so}(2, 4)$ has a Poisson realization on $M = T^*\mathbb{R}_*^3$ from which one can reproduce the Kepler problem — the mathematical model for the simplest solar system. Here $\mathbb{R}_*^3 := \mathbb{R}^3 \setminus \{0\}$ is the configuration space for the Kepler problem. As far as we know, this Poisson realization, more precisely its quantized form, was initially discovered¹ by A.O. Barut and H. Kleinert [1] in 1967.

A discovery made by H. McIntosh and A. Cisneros [2] and independently by D. Zwanziger [3] says that the Kepler problem belongs to a family of dynamic models which share the characteristic feature of the Kepler problem, such as the existence of an analogue of the Laplace-Runge-Lenz vector. These models, referred to as the **MICZ-Kepler problems** (or MIC-Kepler problems) in the literature, are indexed by a real parameter μ (called the magnetic charge) with the Kepler problem corresponding to $\mu = 0$.

Soon after the discovery of the MICZ-Kepler problems, it was realized that the aforementioned Poisson realization of $\mathfrak{so}(2, 4)$ on the phase space of the Kepler problem has

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¹ For the prehistory of this important discovery about the Kepler problem, one may consult Footnote 2 in Ref. [1].

an analogue for each MICZ-Kepler problem. Indeed, an explicit quantized form of these Poisson realizations are given by Eqns (A1) and (A14) in Ref. [4].

In the literature each such individual Poisson realization of $\mathfrak{so}(2, 4)$ is referred to as the **classical dynamic symmetry** for the corresponding MICZ-Kepler problem. Formally, given a dynamic problem P whose phase space is a Poisson manifold M , a Poisson realization \mathcal{R} of certain real Lie algebra \mathfrak{g} on M , if it exists, is called the classical dynamic symmetry for P provided that P and its solutions can be completely derived from \mathcal{R} . In this sense, the classical dynamic symmetry for the isotropic oscillator in dimension n , with $\mathfrak{g} = \mathfrak{sp}_{2n}(\mathbb{R})$, is also known in the literature. More recently, the classical dynamic symmetry for the magnetized Kepler problems in odd dimension $n = 2k + 1$, with $\mathfrak{g} = \mathfrak{so}(2, 2k + 2)$, is explicitly given in Ref. [5].

Very recently the classical $U(1)$ -Kepler problems have been introduced by the second author [6], along with their trajectory analysis via an idea originated from Levi-Civita [7]. This family of models is indexed by two parameters: an integral parameter $n \geq 2$ and a real parameter μ , and its subfamily with $n = 2$ is precisely the family of MICZ-Kepler problems. So it is natural for us to extend the classical dynamic symmetry analysis from the MICZ-Kepler problems to the $U(1)$ -Kepler problems.

In Section 2 we shall give a quick review of Euclidean Jordan algebras [8]. This review is primarily based on the book by J. Faraut and A. Korányi [9]. In Section 3 the classical dynamic symmetry for the Jordan-algebra-based generalized (unmagnetized) Kepler problems [10], is given. In principle, this classical dynamic symmetry can be deduced from its quantized version in Ref. [10], but we shall give it a direct verification. In section 4 we present a family of Poisson realizations for $\mathfrak{su}(n, n)$, summarized in Theorem 2. The proof of this theorem is very length, so Section 5 is devoted to it exclusively. As a consequence of this theorem, we obtain the Laplace-Runge-Lenz vector for any $U(1)$ -Kepler problem. In the final section we describe and prove some quadratic relations concerning this family of Poisson realizations, summarized in Theorem 3. As a corollary of this last theorem, for each $U(1)$ -Kepler problem, we derive a formula connecting its Hamiltonian to its angular momentum and its Laplace-Runge-Lenz vector, generalizing the formula given by Eq. (2.8) of Ref. [11].

2. EUCLIDEAN JORDAN ALGEBRAS

Let V be a simple Euclidean Jordan algebra, which means that V is both a simple Jordan algebra and an Euclidean vector space such that the Jordan multiplication by any element $u \in V$, an endomorphism on V which is denoted by L_u , is self-adjoint with respect to its inner product $\langle | \rangle$. In our convention the identity element e of V is assumed to be a unit vector, so

$$\langle u|v \rangle = \frac{1}{\rho} \text{tr}(uv)$$

where ρ is the rank of V , uv is the Jordan multiplication of u with v , tr means the trace. To say V is a Jordan algebra means that the bilinear map $(u, v) \mapsto uv$ is symmetric and satisfies the Jordan identity: $L_u \circ L_{u^2} = L_{u^2} \circ L_u$ for any $u \in V$.

In the following we shall identify V with V^* via this map:

$$u \in V \mapsto \langle u| \rangle : V \rightarrow \mathbb{R}.$$

We shall use S_{uv} to denote the endomorphism $[L_u, L_v] + L_{uv}$ for each $u, v \in V$. Note that $S_{ue} = S_{eu} = L_u$. Let us denote $S_{uv}(w)$ by $\{uvw\}$, then we have

$$[S_{uv}, S_{zw}] = S_{\{uvz\}w} - S_{z\{vzw\}},$$

so these S_{uv} span a real Lie algebra, referred to as the *structure algebra* for V , and denoted by $\mathfrak{str}(V)$. The *conformal algebra* of V , denoted by $\mathfrak{co}(V)$, is an extension of the structure algebra $\mathfrak{str}(V)$. As a real vector space, we have

$$\mathfrak{co}(V) = V \oplus \mathfrak{str}(V) \oplus V^*.$$

By writing $z \in V$ as X_z , $\langle w | \rangle \in V^*$ as Y_w , the commutation relations on $\mathfrak{co}(V)$ can be written as follow: for u, v, z, w in V ,

$$\begin{cases} [X_u, X_v] = 0, & [Y_u, Y_v] = 0, & [X_u, Y_v] = -2S_{uv}, \\ [S_{uv}, X_z] = X_{\{uvz\}}, & [S_{uv}, Y_z] = -Y_{\{vuz\}}, \\ [S_{uv}, S_{zw}] = S_{\{uvz\}w} - S_{z\{vuw\}}. \end{cases}$$

Note that, when the Jordan algebra is $\Gamma(3): \mathbb{R} \oplus \mathbb{R}^3$ (a linear subspace of the real Clifford algebra $\text{Cl}(\mathbb{R}^3, \text{dot product})$) with the product being the symmetrized Clifford multiplication, we have $\mathfrak{co}(V) = \mathfrak{so}(2, 4)$ — the conformal algebra of the Minkowski space, and $\mathfrak{str}(V) = \mathfrak{so}(1, 3) \oplus \mathbb{R}$. For the case concerning us, $V = H_n(\mathbb{C})$ and $\mathfrak{co}(V) = \mathfrak{su}(n, n)$. Since $H_2(\mathbb{C}) \cong \Gamma(3)$ as Jordan algebra, it is not a surprise that $\mathfrak{su}(2, 2) \cong \mathfrak{so}(2, 4)$ as Lie algebra.

3. THE CLASSICAL DYNAMIC SYMMETRY FOR THE GENERALIZED (UNMAGNETIZED) KEPLER PROBLEMS

In Ref. [10] the second author introduced the Jordan-algebra-based generalized (unmagnetized) Kepler problems, both the quantum models and the classical models. While a few aspects of the quantum models, such as their bound state problem and the quantum dynamic symmetry, are studied in that reference, not a single aspect of the classical models is studied there. In this section we shall devote our attention to the classical dynamic symmetry. This is an easy aspect for the classical models because it can be deduced quickly from its quantum analogue, but the main point here is to set the stage for the study of the classical dynamic symmetry of U(1) Kepler problems as well as the generic (magnetized) Kepler problems in the future.

3.1. Kepler cones. As before V denotes a simple Euclidean Jordan algebra with rank ρ . We shall also consider V as an Euclidean space, i.e., a smooth space (i.e. a manifold) with the Riemannian metric

$$(3.1) \quad ds^2 = \langle dx | dx \rangle.$$

Here x is the identity map on V , but viewed as a map from the smooth space V to the vector space V so that dx , being the total differential of this vector-valued smooth function, is a vector-valued differential one-form on the smooth space V .

For each positive integer k which is at most ρ , we let $\mathcal{C}_k(V)$ or simply \mathcal{C}_k be the set of rank k semi-positive elements of V . It is a fact [10] that \mathcal{C}_k is a submanifold of V and the tangent space of \mathcal{C}_k at a point x is

$$\{x\} \times \text{Im} L_x$$

where $\text{Im}L_x$ denotes the image of the linear map L_x . Moreover, the structure group of V acts on \mathcal{C}_k homogeneously, whose cotangent lift is a symplectic action on $T^*\mathcal{C}_k$. This implies that we have a Poisson realization of the structure algebra \mathfrak{str} on the Poisson manifold $T^*\mathcal{C}_k$. The surprise is that, this Poisson realization of \mathfrak{str} can be extended to a Poisson realization of the conformal algebra \mathfrak{co} .

Before presenting this Poisson realization of \mathfrak{co} on $T^*\mathcal{C}_k$, we need to do some preparations. First of all, $T^*\mathcal{C}_k$ shall be identified with $T\mathcal{C}_k$ via the Riemannian metric (3.1). With this identification understood, $T\mathcal{C}_k$ becomes a Poisson manifold. Next, we write the inclusion map

$$T\mathcal{C}_k \hookrightarrow TV = V \times V$$

as (x, π) , and view both x and π as vector-valued smooth functions on $T\mathcal{C}_k$. Note that, at any point Q of $T\mathcal{C}_k$, $x(Q) \in \mathcal{C}_k$ and $\pi(Q) \in \text{Im}L_{x(Q)}$.

We use q^i to denote a system of local coordinates on \mathcal{C}_k , ∂_{q^i} to denote the resulting local tangent frame, and let

$$g_{ij} := \langle \partial_{q^i} | \partial_{q^j} \rangle, \quad g := [g_{ij}], \quad g^{ij} := (g^{-1})_{ij}, \quad E^i = g^{ij} \partial_{q^j}.$$

Under the identification of $T^*\mathcal{C}_k$ with $T\mathcal{C}_k$ mentioned early, one can see that the local cotangent frame (dq^1, dq^2, \dots) becomes the local tangent frame (E^1, E^2, \dots) , in terms of which we can write

$$\pi = p_i E^i.$$

Also, under the natural identification of $T_x V$ with V , we have $\partial_{q^i} = \frac{\partial x}{\partial q^i}$. Since $\langle E^j | \partial_{q^i} \rangle = dq^j(\partial_{q^i}) = \delta_i^j$, we know that $\langle E^i | v \rangle \partial_{q^i} |_x$ is the orthogonal projection of v onto $\text{Im}L_x$. So, if we denote by \bar{v} the function which maps $x \in \mathcal{C}_k$ to the orthogonal projection of v onto $\text{Im}L_x$, then

$$(3.2) \quad \langle u | \partial_{q^i} \rangle \langle E^i | v \rangle = \langle u | \bar{v} \rangle.$$

For notational sanity, we use the same notation for both a local function on \mathcal{C}_k and its pull-back under the tangent bundle map $\tau: T\mathcal{C}_k \rightarrow \mathcal{C}_k$. For example, q^i denotes both a local function on \mathcal{C}_k and its pullback to $T\mathcal{C}_k$.

Lemma 3.1. *For any vectors $u, v \in V$, viewed as constant vector-valued functions on $T\mathcal{C}_k$, we have*

$$(3.3) \quad \begin{cases} \{\langle u | x \rangle, \langle v | x \rangle\} &= 0, \\ \{\langle u | x \rangle, \langle v | \pi \rangle\} &= \langle u | \bar{v} \rangle, \\ \{\langle u | \pi \rangle, \langle v | \pi \rangle\} &= p_i g^{il} \left\langle \bar{u} \left| \frac{\partial^2 x}{\partial q^j \partial q^l} \right. \right\rangle \langle v | E^j \rangle - \langle u \leftrightarrow v \rangle \end{cases}$$

where $\bar{u} = u - \bar{u}$, and $\langle u \leftrightarrow v \rangle$ denotes the preceding term with u and v being switched. Consequently, for functions \bar{u}, \bar{v} on \mathcal{C}_k whose value at $x \in \mathcal{C}_1$ is inside $\text{Im}L_x$, we have

$$(3.4) \quad \left\{ \langle \bar{u} | \pi \rangle, \langle \bar{v} | \pi \rangle \right\} = 0.$$

Here $\overline{\pi \cdots \pi}$ means only the Poisson bracket between the two π 's is counted.

Proof. The verification of the Poisson bracket relations in Eq. (3.3) is based on the local canonical Poisson bracket relations:

$$(3.5) \quad \{q^i, q^j\} = 0, \quad \{q^i, p_j\} = \delta_j^i, \quad \{p_i, p_j\} = 0.$$

Since x depends on q only, we have

$$\{\langle u|x\rangle, \langle v|x\rangle\} = 0.$$

Next, we have

$$\begin{aligned} \{\langle u|x\rangle, \langle v|\pi\rangle\} &= \{\langle u|x\rangle, p_i \langle v|E^i\rangle\} \\ &= \{\langle u|x\rangle, p_i\} \langle v|E^i\rangle \quad \text{both } x \text{ and } E_i \text{ depend on } q \text{ only} \\ &= \langle u|\partial_{q^i}\rangle \langle E^i|v\rangle \quad \text{using Eq. (3.5)} \\ &= \langle u|\bar{v}\rangle \quad \text{using Eq. (3.2).} \end{aligned}$$

Similarly, we have

$$\begin{aligned} \{\langle u|\pi\rangle, \langle v|\pi\rangle\} &= \{\langle u|p_i E^i\rangle, \langle v|p_j E^j\rangle\} \\ &= p_i \{\langle u|E^i\rangle, p_j\} \langle v|E^j\rangle + \langle u|E^i\rangle \{p_i, \langle v|E^j\rangle\} p_j \\ &= p_i \{\langle u|E^i\rangle, p_j\} \langle v|E^j\rangle - \langle u \leftrightarrow v\rangle. \end{aligned}$$

Since

$$\begin{aligned} \{\langle u|E^i\rangle, p_j\} &= \{\langle u|g^{il}\partial_{q^l}\rangle, p_j\} \\ &= \left\{ g^{il} \left\langle u \left| \frac{\partial x}{\partial q^l} \right\rangle, p_j \right\} \\ &= g^{il} \left\langle u \left| \frac{\partial^2 x}{\partial q^j \partial q^l} \right\rangle + \frac{\partial g^{il}}{\partial q^j} \left\langle u \left| \frac{\partial x}{\partial q^l} \right\rangle \right. \\ &= g^{il} \left\langle u \left| \frac{\partial^2 x}{\partial q^j \partial q^l} \right\rangle - g^{im} g^{nl} \frac{\partial g_{mn}}{\partial q^j} \left\langle u \left| \frac{\partial x}{\partial q^l} \right\rangle \right. \\ &= g^{il} \left\langle u \left| \frac{\partial^2 x}{\partial q^j \partial q^l} \right\rangle - g^{im} \frac{\partial g_{mn}}{\partial q^j} \langle u|E^n\rangle \\ &= g^{il} \left\langle u \left| \frac{\partial^2 x}{\partial q^j \partial q^l} \right\rangle - g^{im} \partial_{q^j} \left(\left\langle \frac{\partial x}{\partial q^m} \left| \frac{\partial x}{\partial q^n} \right\rangle \right) \langle u|E^n\rangle \right. \\ &= g^{il} \left\langle u \left| \frac{\partial^2 x}{\partial q^j \partial q^l} \right\rangle - g^{im} \left\langle \frac{\partial^2 x}{\partial q^j \partial q^m} \left| \bar{u} \right\rangle - \left\langle E^i \left| \frac{\partial^2 x}{\partial q^j \partial q^n} \right\rangle \langle u|E^n\rangle \right. \\ &= g^{il} \left\langle \bar{u} \left| \frac{\partial^2 x}{\partial q^j \partial q^l} \right\rangle - \left\langle E^i \left| \frac{\partial^2 x}{\partial q^j \partial q^n} \right\rangle \langle u|E^n\rangle, \end{aligned}$$

we have

$$\begin{aligned} \{\langle u|\pi\rangle, \langle v|\pi\rangle\} &= p_i \{\langle u|E^i\rangle, p_j\} \langle v|E^j\rangle - \langle u \leftrightarrow v\rangle \\ &= p_i \left(g^{il} \left\langle \bar{u} \left| \frac{\partial^2 x}{\partial q^j \partial q^l} \right\rangle - \left\langle E^i \left| \frac{\partial^2 x}{\partial q^j \partial q^n} \right\rangle \langle u|E^n\rangle \right) \langle v|E^j\rangle - \langle u \leftrightarrow v\rangle \right. \\ &= p_i g^{il} \left\langle \bar{u} \left| \frac{\partial^2 x}{\partial q^j \partial q^l} \right\rangle \langle v|E^j\rangle - \langle u \leftrightarrow v\rangle. \end{aligned}$$

□

3.2. The classical dynamic symmetry. We are now ready to state the Poisson realization of the conformal algebra \mathfrak{co} on TC_k — the dynamical symmetry for the generalized Kepler problem with configuration space \mathcal{C}_k .

Theorem 1. *For any vectors u, v in V , define functions*

$$(3.6) \quad \mathcal{X}_u := \langle x | \{\pi u \pi\} \rangle, \quad \mathcal{S}_{uv} := \langle S_{uv}(x) | \pi \rangle, \quad \mathcal{Y}_v := \langle v | x \rangle$$

on TC_k . Then, for any vectors u, v, z, w in V , the following Poisson bracket relations hold:

$$\begin{cases} \{\mathcal{X}_u, \mathcal{X}_v\} = 0, & \{\mathcal{Y}_u, \mathcal{Y}_v\} = 0, & \{\mathcal{X}_u, \mathcal{Y}_v\} = -2\mathcal{S}_{uv}, \\ \{\mathcal{S}_{uv}, \mathcal{X}_z\} = \mathcal{X}_{\{uvz\}}, & \{\mathcal{S}_{uv}, \mathcal{Y}_z\} = -\mathcal{Y}_{\{vuz\}}, \\ \{\mathcal{S}_{uv}, \mathcal{S}_{zw}\} = \mathcal{S}_{\{uvz\}w} - \mathcal{S}_{z\{vuw\}}. \end{cases}$$

Proof. The proof is a straightforward computation based on the Poisson bracket relations in Eq. (3.3). Due to the Poisson bracket relation in Eq. (3.4), this proof is really a verbatim copy of the proof for Theorem 3.1 in [10], so it is skipped here. \square

Remark 3.1. The generalized Kepler problem corresponding to the Poisson realization in Theorem 1 is the Hamiltonian system with phase space TC_k , Hamiltonian

$$H = \frac{1}{2} \frac{\mathcal{X}_e}{\mathcal{Y}_e} - \frac{1}{\mathcal{Y}_e},$$

and Laplace-Runge-Lenz vector

$$\mathcal{A}_u = \frac{1}{2} \left(\mathcal{X}_u - \mathcal{Y}_u \frac{\mathcal{X}_e}{\mathcal{Y}_e} \right) + \frac{\mathcal{Y}_u}{\mathcal{Y}_e}.$$

The interested readers may consult Ref. [12] for more details on this point.

The following subsection is a detailed demonstration of this remark for the Kepler problem.

3.3. Example: Kepler problem and future light-cone. The purpose here is to show explicitly a claim made by the 2nd author in the past: if $V = \Gamma(3) := \mathbb{R} \oplus \mathbb{R}^3$, and $k = 1$, the generalized Kepler problem is exactly the Kepler problem. In terms of the standard basis vectors $\vec{e}_0, \vec{e}_1, \vec{e}_2, \vec{e}_3$, the Jordan multiplication can be determined by the following rules: \vec{e}_0 is the identity element, and

$$\vec{e}_i \vec{e}_j = \delta_{ij} \vec{e}_0$$

for $i, j > 0$. The trace $\text{tr} : V \rightarrow \mathbb{R}$ is given by the following rules:

$$\text{tr } \vec{e}_0 = 2, \quad \text{tr } \vec{e}_i = 0.$$

So the inner product on V is the one such that the standard basis is an orthonormal basis. Since V has rank two, the determinant of $x = x^\mu \vec{e}_\mu$ is

$$\det x = \frac{1}{2} ((\text{tr } x)^2 - \text{tr } x^2) = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2.$$

Therefore,

$$\mathcal{C}_1 = \{x \in V \mid \det x = 0, \text{tr } x > 0\}$$

is precisely the future light-cone in the Minkowski space. It turns out that \mathcal{C}_1 has a global coordinate $q = (q^1, q^2, q^3)$ with $q^i(x) = x^i$. Since $x(q) = r\vec{e}_0 + \vec{r}$ where $\vec{r} = q^i \vec{e}_i$ and r is the length of \vec{r} , we have

$$\partial_{q^i} = \vec{e}_i + \frac{q^i}{r} \vec{e}_0, \quad g_{ij} = \delta_{ij} + \frac{q^i q^j}{r^2}, \quad g^{ij} = \delta_{ij} - \frac{q^i q^j}{2r^2}, \quad E^j = \vec{e}_j - \frac{q^j}{2r^2} \vec{r} + \frac{q^j}{2r} \vec{e}_0.$$

Here the first and last identities are understood with the natural identification of $T_x \mathcal{C}_1$ with $\text{Im} L_x$ in mind.

Let $\vec{p} = \sum_i p_i \vec{e}_i$ and $|\vec{p}|^2 = \vec{p} \cdot \vec{p}$. Since $\pi = p_i E^i = \vec{p} - \frac{\vec{p} \cdot \vec{r}}{2r^2} \vec{r} + \frac{\vec{p} \cdot \vec{r}}{2r} \vec{e}_0$, then $x\pi = (\vec{p} \cdot \vec{r}) \vec{e}_0 + r\vec{p}$, therefore

$$\begin{aligned} \mathcal{X}_e &= \langle x | \pi^2 \rangle = \langle x\pi | \pi \rangle \\ &= \left\langle (\vec{p} \cdot \vec{r}) \vec{e}_0 + r\vec{p} \left| \vec{p} - \frac{\vec{p} \cdot \vec{r}}{2r^2} \vec{r} + \frac{\vec{p} \cdot \vec{r}}{2r} \vec{e}_0 \right. \right\rangle \\ &= r|\vec{p}|^2. \end{aligned}$$

Since $\mathcal{Y}_e = r$, we have the Hamiltonian

$$H = \frac{1}{2} \frac{\mathcal{X}_e}{\mathcal{Y}_e} - \frac{1}{\mathcal{Y}_e} = \frac{1}{2} |\vec{p}|^2 - \frac{1}{r}.$$

Similarly, one can compute $\mathcal{X}_{\vec{e}_i} = \langle x | \{\pi \vec{e}_i \pi\} \rangle$ and $\mathcal{Y}_{\vec{e}_i} = \langle x | \vec{e}_i \rangle$ and arrive at

$$\sum_i \mathcal{X}_{\vec{e}_i} \vec{e}_i = 2(\vec{r} \cdot \vec{p}) \vec{p} - \vec{r} |\vec{p}|^2, \quad \sum_i \mathcal{Y}_{\vec{e}_i} \vec{e}_i = \vec{r}.$$

Then we arrive at the usual Lapace-Runge-Lenz vector for the Kepler problem:

$$\begin{aligned} \vec{A} &:= \sum_i A_{\vec{e}_i} \vec{e}_i = \sum_i \left(\frac{1}{2} \left(\mathcal{X}_{\vec{e}_i} - \mathcal{Y}_{\vec{e}_i} \frac{\mathcal{X}_e}{\mathcal{Y}_e} \right) + \frac{\mathcal{Y}_{\vec{e}_i}}{\mathcal{Y}_e} \right) \vec{e}_i \\ &= (\vec{r} \times \vec{p}) \times \vec{p} + \frac{\vec{r}}{r}. \end{aligned}$$

Remark 3.2. *A far as we know, the fact that the Laplace-Runge-Lenz vector owes its existence to the dynamic symmetry was initially pointed out by the second author in Subsection 7.1 of Ref. [13].*

4. THE CLASSICAL DYNAMICAL SYMMETRY FOR THE U(1) KEPLER PROBLEMS

In the remainder of this article the simple Euclidean Jordan algebra V is assumed to be $H_n(\mathbb{C})$ — the Jordan algebra of complex hermitian matrices of order $n \geq 2$, and μ is assumed to be a real number.

In this case \mathcal{C}_1 is homotopy equivalent to $\mathbb{C}P^{n-1}$, and a generator of $H^2(\mathcal{C}_1, \mathbb{Z}) \cong \mathbb{Z}$ can be chosen to be the cohomology class of the closed real differential two-form $\frac{\omega_K}{2\pi}$ where

$$(4.1) \quad \omega_K := -i \frac{\text{tr}(x \, dx \wedge dx)}{(\text{tr } x)^3}$$

and is called the Kepler form in Ref. [6]. On a topologically trivial coordinate patch, there is a real differential one-form $A = A_i dq^i$ such that

$$\omega_K = dA.$$

We shall also use ω_K to denote the pullback of ω_K under the cotangent bundle projection map $T^*\mathcal{C}_1 \rightarrow \mathcal{C}_1$. Let $\omega_{\mathcal{C}_1}$ be the canonical symplectic form on $T^*\mathcal{C}_1$ and

$$\omega_\mu := \omega_{\mathcal{C}_1} + 2\mu \omega_K.$$

On a topologically trivial coordinate patch we have

$$\omega_\mu = dp_i \wedge dq^i + 2\mu dA = d(p_i + 2\mu A_i) \wedge dq^i,$$

we conclude that ω_μ is a symplectic form on $T^*\mathcal{C}_1$. As before we shall identify $T^*\mathcal{C}_1$ with TC_1 via the inner product on V . In this case elements in V are hermitian matrices, so, for any $u, v \in V$, we have matrix product $u \cdot v$, in terms of which, we have the commutator $[u, v] = u \cdot v - v \cdot u$ and the Jordan product $uv = \frac{1}{2}(u \cdot v + v \cdot u)$.

Lemma 4.1. *Assume that $x \in \mathcal{C}_1$. Let $u, v \in H_n(\mathbb{C})$ and $L_{u,v} = [L_u, L_v]$.*

(i) We have $x^2 = \text{tr } x \, x$, consequently

$$[x, ux] = \frac{\text{tr } x}{2}[x, u], \quad \text{tr}(x[ux, v]) = \frac{1}{2}\text{tr } x \, \text{tr}(x[u, v]).$$

(ii) The following identities hold:

$$i[u, x] \in \text{Im } L_x, \quad x \cdot u \cdot x = \text{tr}(xu) \, x, \quad L_{u,v}x = \frac{1}{4}[[u, v], x].$$

Consequently, we have

$$(4.2) \quad \frac{1}{2}[x, [x, u]] = (\text{tr } x)xu - \text{tr}(xu)x.$$

(iii) The following Poisson bracket relations on the symplectic manifold (TC_1, ω_μ) hold:

$$\{\langle u|x\rangle, \langle v|x\rangle\} = 0, \quad \{\langle u|x\rangle, \langle v|\pi\rangle\} = \langle u|\bar{v}\rangle,$$

and

$$\left\{ \langle \tilde{u}|\pi\rangle, \overline{\langle \tilde{v}|\pi\rangle} \right\} = -2\mu i \frac{\text{tr}(x[\tilde{u}, \tilde{v}])}{(\text{tr } x)^3}$$

provided that \tilde{u}, \tilde{v} are functions on \mathcal{C}_1 whose value at $x \in \mathcal{C}_1$ is inside $\text{Im } L_x$.

Proof. (i) Since x has rank 1 and is diagonalizable, it is clear that $x^2 = \text{tr } x \, x$. Then

$$\begin{aligned} [x, ux] &= \frac{1}{2}(x \cdot (u \cdot x + x \cdot u) - (u \cdot x + x \cdot u) \cdot x) \\ &= \frac{1}{2}(x^2 \cdot u - u \cdot x^2) = \frac{\text{tr } x}{2}[x, u] \end{aligned}$$

and

$$\begin{aligned} \text{tr}(x[ux, v]) &= \frac{1}{2}\text{tr}(x \cdot (u \cdot x \cdot v + x \cdot u \cdot v - v \cdot x \cdot u - v \cdot u \cdot x)) \\ &= \frac{1}{2}\text{tr}(x^2 \cdot u \cdot v - x^2 \cdot v \cdot u) \quad \text{using the fact that tr is cyclic} \\ &= \frac{1}{2}\text{tr } x \, \text{tr}(x[u, v]). \end{aligned}$$

(ii) We may assume that x is the matrix whose $(1, 1)$ -entry is 1 and all other entries are zero. Then $\text{Im } L_x$ is the set consisting of hermitian matrices whose (i, j) -entry is zero if $i, j > 1$. It is then clear that the hermitian matrix $i[u, x]$ is an element of $\text{Im } L_x$.

It is clear that $x \cdot u \cdot x$ is the matrix whose $(1, 1)$ -entry is u_{11} (the $(1, 1)$ -entry of u) and all other entries are zero. Since $\text{tr}(xu) = u_{11}$, we have $x \cdot u \cdot x = \text{tr}(xu) \, x$. Consequently

$$\begin{aligned} \frac{1}{2}[x, [x, u]] &= \frac{1}{2}(x \cdot (x \cdot u - u \cdot x) - (x \cdot u - u \cdot x) \cdot x) \\ &= x^2 u - x \cdot u \cdot x \\ &= (\text{tr } x)xu - \text{tr}(xu)x. \end{aligned}$$

The identity $L_{u,v}x = \frac{1}{4}[[u, v], x]$ actually holds for any x, u, v :

$$\begin{aligned} L_{u,v}x &= u(vx) - \langle u \leftrightarrow v \rangle \\ &= \frac{1}{4}(u \cdot (v \cdot x + x \cdot v) + (v \cdot x + x \cdot v) \cdot u - \langle u \leftrightarrow v \rangle) \\ &= \frac{1}{4}(u \cdot v \cdot x + u \cdot x \cdot v + v \cdot x \cdot u + x \cdot v \cdot u - \langle u \leftrightarrow v \rangle) \\ &= \frac{1}{4}([u, v] \cdot x + x \cdot [v, u]) \\ &= \frac{1}{4}[[u, v], x]. \end{aligned}$$

- (iii) The proof is similar to the proof of Lemma 3.1. Note that, the local canonical Poisson relations (3.5) is now changed to

$$\{q^i, q^j\} = 0, \quad \{q^i, p_j\} = \delta_j^i, \quad \{p_i, p_j\} = -2\mu i \frac{\text{tr} \left(x \left[\frac{\partial x}{\partial q^i}, \frac{\partial x}{\partial q^j} \right] \right)}{(\text{tr } x)^3}.$$

So the proof of identities $\{\langle u|x\rangle, \langle v|x\rangle\} = 0$ and $\{\langle u|x\rangle, \langle v|\pi\rangle\} = \langle u|\bar{v}\rangle$ is the same as before.

Proof of identity $\left\{ \langle \tilde{u}|\pi\rangle, \langle \tilde{v}|\pi\rangle \right\} = -2\mu i \frac{\text{tr}(x[\tilde{u}, \tilde{v}])}{(\text{tr } x)^3}.$

$$\begin{aligned} LHS &= \langle \tilde{u}|E^i\rangle \langle \tilde{v}|E^j\rangle \{p_i, p_j\} + \text{other terms} \\ &= \langle \tilde{u}|E^i\rangle \langle \tilde{v}|E^j\rangle \{p_i, p_j\} + 0 \quad \text{using Eq. (3.4)} \\ &= \langle \tilde{u}|E^i\rangle \langle \tilde{v}|E^j\rangle (-2\mu i) \frac{\text{tr} \left(x \left[\frac{\partial x}{\partial q^i}, \frac{\partial x}{\partial q^j} \right] \right)}{(\text{tr } x)^3} \\ &= -2\mu i \frac{\text{tr}(x[\tilde{u}, \tilde{v}])}{(\text{tr } x)^3} \\ &= RHS. \end{aligned}$$

□

Theorem 2. For any vectors u, v in $V := H_n(\mathbb{C})$, define functions

$$(4.3) \quad \begin{cases} \mathcal{X}_u &:= \langle x|\{\pi u \pi\}\rangle + \frac{n\mu^2}{(\text{tr } x)^2} \text{tr}(xu) - \mu i \frac{\text{tr}(x[u, \pi])}{\text{tr } x} \\ \mathcal{Y}_v &:= \langle v|x\rangle \\ \mathcal{S}_{uv} &:= \langle S_{uv}(x)|\pi\rangle - \mu i \frac{\text{tr}(x[u, v])}{2 \text{tr } x} \end{cases}$$

on TC_1 . Then, for any vectors u, v, z, w in V , the following Poisson bracket relations hold:

$$\begin{cases} \{\mathcal{X}_u, \mathcal{X}_v\} = 0, & \{\mathcal{Y}_u, \mathcal{Y}_v\} = 0, & \{\mathcal{X}_u, \mathcal{Y}_v\} = -2\mathcal{S}_{uv}, \\ \{\mathcal{S}_{uv}, \mathcal{X}_z\} = \mathcal{X}_{\{uvz\}}, & \{\mathcal{S}_{uv}, \mathcal{Y}_z\} = -\mathcal{Y}_{\{vuz\}}, \\ \{\mathcal{S}_{uv}, \mathcal{S}_{zw}\} = \mathcal{S}_{\{uvz\}w} - \mathcal{S}_{z\{vuw\}}. \end{cases}$$

The proof of this theorem is a bit complicated, so we leave it to the next section.

Remark 4.1. In view of Ref. [12], Theorem 2 implies that the corresponding $U(1)$ Kepler problem is the Hamiltonian system with phase space TC_1 , Hamiltonian

$$H = \frac{1}{2} \frac{\mathcal{X}_e}{\mathcal{Y}_e} - \frac{1}{\mathcal{Y}_e}$$

and Laplace-Runge-Lenz vector

$$\mathcal{A}_u = \frac{1}{2} \left(\mathcal{X}_u - \mathcal{Y}_u \frac{\mathcal{X}_e}{\mathcal{Y}_e} \right) + \frac{\mathcal{Y}_u}{\mathcal{Y}_e}.$$

A simple computation yields $H = \frac{\langle x|\pi^2\rangle}{2r} + \frac{n^2\mu^2}{2(\text{tr } x)^2} - \frac{n}{\text{tr } x}$, i.e., the Hamiltonian in Definition 1.1 of Ref. [6].

5. PROOF OF THEOREM 2

The proof is heavily dependent on Lemma 4.1. Theorem 1 says that these identities hold for the constant terms in μ , so we just need to verify them at higher order terms in μ .

Step zero: It is clear that $\{\mathcal{Y}_u, \mathcal{Y}_v\} = 0$.

Step one: Verify that $\{\mathcal{S}_{uv}, \mathcal{Y}_z\} = -\mathcal{Y}_{\{vuz\}}$. This is easy:

$$\begin{aligned} \{\mathcal{S}_{uv}, \mathcal{Y}_z\} &= \{\langle S_{uv}(x)|\pi\rangle - \mu i \frac{\text{tr}(x[u, v])}{2 \text{tr } x}, \langle z|x\rangle\} \\ &= \{\langle S_{uv}(x)|\pi\rangle, \langle z|x\rangle\} \\ &= -\mathcal{Y}_{\{vuz\}} \quad \text{no higher order terms in } \mu \text{ involved here.} \end{aligned}$$

Step two: Verify that $\{\mathcal{X}_u, \mathcal{Y}_v\} = -2\mathcal{S}_{uv}$.

$$\begin{aligned} \{\mathcal{X}_u, \mathcal{Y}_v\} &= \{\langle x|\{\pi u \pi\}\rangle + \frac{n\mu^2}{(\text{tr } x)^2} \text{tr}(xu) - \mu i \frac{\text{tr}(x[u, \pi])}{\text{tr } x}, \langle v|x\rangle\} \\ &= -2\langle S_{uv}(x)|\pi\rangle - \frac{\mu i}{\text{tr } x} \{\text{tr}(x[u, \pi]), \langle v|x\rangle\} \\ &= -2\langle S_{uv}(x)|\pi\rangle + \frac{\mu}{\text{tr } x} \{\langle v|x\rangle, \text{tr}(i[x, u]\pi)\} \\ &= -2\langle S_{uv}(x)|\pi\rangle + \frac{\mu}{\text{tr } x} \text{tr}(i[x, u]v) \quad \text{using Lemma 4.1} \\ &= -2\langle S_{uv}(x)|\pi\rangle + \mu i \frac{\text{tr}(x[u, v])}{\text{tr } x} \\ &= -2\mathcal{S}_{uv}. \end{aligned}$$

Step three: Verify that $\{\mathcal{S}_{uv}, \mathcal{S}_{zw}\} = \mathcal{S}_{\{uvz\}w} - \mathcal{S}_{z\{vuw\}}$. This is a bit involved.

Let $\mathcal{L}_u := \mathcal{S}_{ue}$ and $\mathcal{L}_{u,v} := \frac{1}{2}(\mathcal{S}_{uv} - \mathcal{S}_{vu})$. Then

$$\mathcal{L}_u = \langle ux|\pi\rangle, \quad \mathcal{L}_{u,v} = \langle L_{u,v}x|\pi\rangle - \mu i \frac{\text{tr}(x[u, v])}{2 \text{tr } x}, \quad \mathcal{S}_{uv} = \mathcal{L}_{u,v} + \mathcal{L}_{uv}.$$

We claim that

$$(5.1) \quad \{\mathcal{L}_u, \mathcal{L}_v\} = \mathcal{L}_{u,v}, \quad \{\mathcal{L}_{u,v}, \mathcal{L}_z\} = \mathcal{L}_{L_{u,v}z}.$$

Proof that $\{\mathcal{L}_u, \mathcal{L}_v\} = \mathcal{L}_{u,v}$:

$$\begin{aligned} \{\mathcal{L}_u, \mathcal{L}_v\} &= \{\langle ux|\pi\rangle, \langle vx|\pi\rangle\} \\ &= \langle L_{u,v}x|\pi\rangle + \{\langle ux|\pi\rangle, \langle vx|\pi\rangle\} \\ &= \langle L_{u,v}x|\pi\rangle - 2\mu i \frac{\text{tr}(x[ux, vx])}{(\text{tr } x)^3} \\ &= \langle L_{u,v}x|\pi\rangle - \mu i \frac{\text{tr}(x[u, v])}{2 \text{tr } x} \\ &= \mathcal{L}_{u,v}. \end{aligned}$$

Proof that $\{\mathcal{L}_{u,v}, \mathcal{L}_z\} = \mathcal{L}_{L_{u,v}z}$:

$$\begin{aligned} \{\mathcal{L}_{u,v}, \mathcal{L}_z\} &= \{\langle L_{u,v}x|\pi\rangle - \mu i \frac{\text{tr}(x[u, v])}{2 \text{tr } x}, \langle zx|\pi\rangle\} \\ &= \mathcal{L}_{L_{u,v}z} + \mu i \left\{ \langle zx|\pi\rangle, \frac{\text{tr}(x[u, v])}{2 \text{tr } x} \right\} + \{\langle L_{u,v}x|\pi\rangle, \langle zx|\pi\rangle\} \\ &= \mathcal{L}_{L_{u,v}z} + \mu i \left\{ \langle zx|\pi\rangle, \frac{\text{tr}(x[u, v])}{2 \text{tr } x} \right\} - 2\mu i \frac{\text{tr}(x[L_{u,v}x, zx])}{(\text{tr } x)^3} \end{aligned}$$

$$\begin{aligned}
&= \mathcal{L}_{L_{u,v}z} + \mu i \left(-\frac{\text{tr}((zx)[u,v])}{2\text{tr } x} + \frac{\text{tr}(x[u,v])}{2(\text{tr } x)^2} \text{tr}(zx) \right) \\
&\quad - \mu i \frac{\text{tr}(x[L_{u,v}x, z])}{(\text{tr } x)^2} \\
&= \mathcal{L}_{L_{u,v}z}
\end{aligned}$$

provided that

$$-\text{tr}((zx)[u,v])\text{tr } x + \text{tr}(x[u,v])\text{tr}(zx) - 2\text{tr}(x[L_{u,v}x, z]) = 0,$$

or

$$-(\text{tr } x)x[u,v] + \text{tr}(x[u,v])x = 2[x, L_{u,v}x],$$

which is implied by the following identities

$$L_{u,v}x = \frac{1}{4}[[u,v], x], \quad \frac{1}{2}[x, [x, u]] = (\text{tr } x)xu - \text{tr}(xu)x$$

in part (ii) of Lemma 4.1.

We are now ready to prove that $\{\mathcal{S}_{uv}, \mathcal{S}_{zw}\} = \mathcal{S}_{\{uvz\}w} - \mathcal{S}_{z\{vuw\}}$: Since $\mathcal{S}_{uv} = \mathcal{L}_{u,v} + \mathcal{L}_{uv}$ and $\mathcal{S}_{zw} = \mathcal{L}_{z,w} + \mathcal{L}_{zw}$, we have

$$\begin{aligned}
\{\mathcal{S}_{uv}, \mathcal{S}_{zw}\} &= \{\mathcal{L}_{u,v}, \mathcal{L}_{z,w}\} + \{\mathcal{L}_{u,v}, \mathcal{L}_{zw}\} + \{\mathcal{L}_{uv}, \mathcal{L}_{z,w}\} + \{\mathcal{L}_{uv}, \mathcal{L}_{zw}\} \\
&= \{\mathcal{L}_{u,v}, \{\mathcal{L}_z, \mathcal{L}_w\}\} + \mathcal{L}_{L_{u,v}(zw)} - \mathcal{L}_{L_{z,w}(uv)} + \mathcal{L}_{uv,zw} \\
&= \{\{\mathcal{L}_{u,v}, \mathcal{L}_z\}, \mathcal{L}_w\} + \{\mathcal{L}_z, \{\mathcal{L}_{u,v}, \mathcal{L}_w\}\} + \mathcal{L}_{L_{u,v}(zw)} - \mathcal{L}_{L_{z,w}(uv)} + \mathcal{L}_{uv,zw} \\
&= \{\mathcal{L}_{L_{u,v}z}, \mathcal{L}_w\} + \{\mathcal{L}_z, \mathcal{L}_{L_{u,v}w}\} + \mathcal{L}_{L_{u,v}(zw)} - \mathcal{L}_{L_{z,w}(uv)} + \mathcal{L}_{uv,zw} \\
&= \mathcal{L}_{L_{u,v}z,w} + \mathcal{L}_{z,L_{u,v}w} + \mathcal{L}_{L_{u,v}(zw)} - \mathcal{L}_{L_{z,w}(uv)} + \mathcal{L}_{uv,zw} \\
&= \mathcal{L}_{\{uvz\},w} - \mathcal{L}_{(uv)z,w} - \mathcal{L}_{z,\{vuw\}} + \mathcal{L}_{z,(uv)w} \\
&\quad + \mathcal{L}_{L_{u,v}(zw)} - \mathcal{L}_{L_{z,w}(uv)} + \mathcal{L}_{uv,zw} \\
&= \mathcal{S}_{\{uvz\}w} - \mathcal{S}_{z\{vuw\}} - \mathcal{L}_{(uv)z,w} + \mathcal{L}_{z,(uv)w} + \mathcal{L}_{uv,zw} \\
&\quad - \mathcal{L}_{\{uvz\}w} + \mathcal{L}_{z\{vuw\}} + \mathcal{L}_{L_{u,v}(zw)} - \mathcal{L}_{L_{z,w}(uv)} \\
&= \mathcal{S}_{\{uvz\}w} - \mathcal{S}_{z\{vuw\}}
\end{aligned}$$

provided that

$$\mu i \frac{\text{tr}(x[(uv)z, w])}{2\text{tr } x} - \mu i \frac{\text{tr}(x[z, (uv)w])}{2\text{tr } x} - \mu i \frac{\text{tr}(x[uv, zw])}{2\text{tr } x} = 0$$

or

$$\text{tr}(x[(uv)z, w]) - \text{tr}(x[z, (uv)w]) - \text{tr}(x[uv, zw]) = 0$$

or

$$[x, (uv)z] = [x, z](uv) + [x, uv]z$$

which is clearly true because $[x, \cdot]$ is a derivation.

Step four: Verify that $\{\mathcal{S}_{uv}, \mathcal{X}_z\} = \mathcal{X}_{\{uvz\}}$. It suffices to verify that $\{\mathcal{L}_u, \mathcal{X}_v\} = \mathcal{X}_{uv}$:

$$\begin{aligned}
\{\mathcal{S}_{uv}, \mathcal{X}_z\} &= \{\mathcal{L}_{u,v}, \mathcal{X}_z\} + \{\mathcal{L}_{uv}, \mathcal{X}_z\} \\
&= \{\{\mathcal{L}_u, \mathcal{L}_v\}, \mathcal{X}_z\} + \{\mathcal{L}_{uv}, \mathcal{X}_z\} \\
&= \{\{\mathcal{L}_u, \mathcal{X}_z\}, \mathcal{L}_v\} + \{\mathcal{L}_u, \{\mathcal{L}_v, \mathcal{X}_z\}\} + \mathcal{X}_{(uv)z} \\
&= -\mathcal{X}_{v(uz)} + \mathcal{X}_{u(vz)} + \mathcal{X}_{(uv)z}
\end{aligned}$$

$$= \mathcal{X}_{\{uvz\}}.$$

Proof that $\{\mathcal{L}_u, \mathcal{X}_v\} = \mathcal{X}_{uv}$, i.e.,

$$\left\{ \langle ux|\pi\rangle, \langle x|\{\pi v\pi\}\rangle + \frac{n\mu^2}{(\text{tr } x)^2} \text{tr}(xv) - \mu i \frac{\text{tr}(x[v, \pi])}{\text{tr } x} \right\}$$

is equal to

$$\langle x|\{\pi(uv)\pi\}\rangle + \frac{n\mu^2}{(\text{tr } x)^2} \text{tr}(x(uv)) - \mu i \frac{\text{tr}(x[uv, \pi])}{\text{tr } x}$$

which involve terms up to degree two in μ . Note that there is no need to verify it for terms constant in μ because of Theorem 1.

For terms quadratic in μ , we have to verify that

$$\left\{ \langle ux|\pi\rangle, \frac{n\mu^2}{(\text{tr } x)^2} \text{tr}(xv) \right\} - \frac{\mu i}{\text{tr } x} \left\{ \langle ux|\overline{\pi}, \overline{\text{tr}(x[v, \pi])} \right\} = \frac{n\mu^2}{(\text{tr } x)^2} \text{tr}(x(uv)),$$

i.e.,

$$-\frac{n\mu^2}{(\text{tr } x)^2} \text{tr}((ux)v) + 2\frac{n\mu^2}{(\text{tr } x)^3} \text{tr}(xv) \text{tr}(ux) - \frac{n\mu i}{\text{tr } x} \left\{ \langle ux|\overline{\pi}, \overline{\langle [x, v]|\pi\rangle} \right\} = \frac{n\mu^2}{(\text{tr } x)^2} \text{tr}(x(uv))$$

or

$$-\frac{n\mu^2}{(\text{tr } x)^2} \text{tr}((ux)v) + 2\frac{n\mu^2}{(\text{tr } x)^3} \text{tr}(xv) \text{tr}(ux) - \frac{2n\mu^2}{\text{tr } x} \frac{\text{tr}(x[ux, [x, v]])}{(\text{tr } x)^3} = \frac{n\mu^2}{(\text{tr } x)^2} \text{tr}(x(uv))$$

or

$$-\frac{n\mu^2}{(\text{tr } x)^2} \text{tr}((ux)v) + 2\frac{n\mu^2}{(\text{tr } x)^3} \text{tr}(xv) \text{tr}(ux) - \frac{n\mu^2}{(\text{tr } x)^3} \text{tr}(x[u, [x, v]]) = \frac{n\mu^2}{(\text{tr } x)^2} \text{tr}(x(uv))$$

or

$$(5.2) \quad -\text{tr}(v(ux)) + \frac{2}{\text{tr } x} \text{tr}(vx) \text{tr}(ux) - \frac{\text{tr}(x[u, [x, v]])}{\text{tr } x} = \text{tr}(x(uv))$$

or

$$(5.3) \quad -vx + \frac{2}{\text{tr } x} \text{tr}(vx)x - \frac{1}{\text{tr } x} [[x, v], x] = xv$$

which is essentially identity (4.2).

For terms linear in μ , we have to verify that

$$2 \left\{ \langle ux|\overline{\pi}, \overline{\langle x|\{\pi v\pi\}\rangle} \right\} - \mu i \left\{ \langle ux|\pi\rangle, \frac{\text{tr}(x[v, \pi])}{\text{tr } x} \right\} \Big|_{\text{no } \pi\pi \text{ contraction}} = -\mu i \frac{\text{tr}(x[uv, \pi])}{\text{tr } x},$$

i.e.

$$\begin{aligned} -\mu i \frac{\text{tr}(x[(uv), \pi])}{\text{tr } x} &= -4\mu i \frac{\text{tr}(x[ux, S_{v\pi}(x)])}{(\text{tr } x)^3} - \mu i \frac{\text{tr}((u\pi)[x, v])}{\text{tr } x} \\ &\quad + \mu i \frac{\text{tr}((ux)[v, \pi])}{\text{tr } x} - \mu i \frac{\text{tr}(x[v, \pi])}{(\text{tr } x)^2} \text{tr}(ux) \end{aligned}$$

or

$$\begin{aligned} -\mu i \frac{\text{tr}(x[(uv), \pi])}{\text{tr } x} &= -2\mu i \frac{\text{tr}(x[u, S_{v\pi}(x)])}{(\text{tr } x)^2} - \mu i \frac{\text{tr}((u\pi)[x, v])}{\text{tr } x} \\ &\quad + \mu i \frac{\text{tr}((ux)[v, \pi])}{\text{tr } x} - \mu i \frac{\text{tr}(x[v, \pi])}{(\text{tr } x)^2} \text{tr}(ux) \end{aligned}$$

or

$$2\text{tr}(x[u, S_{v\pi}(x)]) = -\text{tr}((u\pi)[x, v]) \text{tr } x + \text{tr}((ux)[v, \pi]) \text{tr } x$$

$$-\text{tr}(x[v, \pi])\text{tr}(ux) + \text{tr}(x[uv, \pi])\text{tr} x$$

or

$$2[S_{v\pi}(x), x] = -\pi[x, v]\text{tr} x + x[v, \pi]\text{tr} x - \text{tr}(x[v, \pi])x + v[\pi, x]\text{tr} x.$$

Expanding the term on the left and combining the 1st and last terms on the right, we arrive at the identity

$$2[v(\pi x), x] + 2[(v\pi)x, x] - 2[\pi(vx), x] = [\pi v, x]\text{tr} x + x[v, \pi]\text{tr} x - \text{tr}(x[v, \pi])x.$$

Since $2[(v\pi)x, x] = [v\pi, x]\text{tr} x$, the preceding identity becomes

$$2[L_{v, \pi}x, x] = x[v, \pi]\text{tr} x - \text{tr}(x[v, \pi])x$$

or

$$\frac{1}{2}[x, [x, [v, \pi]]] = x[v, \pi]\text{tr} x - \text{tr}(x[v, \pi])x$$

which is implied by the following identities

$$L_{u, v}x = \frac{1}{4}[[u, v], x], \quad \frac{1}{2}[x, [x, u]] = (\text{tr} x)xu - \text{tr}(xu)x$$

in part (ii) of Lemma 4.1.

Step five: Verify that $\{\mathcal{X}_u, \mathcal{X}_v\} = 0$. It suffices to verify that $\{\mathcal{X}_u, \mathcal{X}_e\} = 0$:

$$\begin{aligned} \{\mathcal{X}_u, \mathcal{X}_v\} &= \{\mathcal{X}_u, \{\mathcal{L}_v, \mathcal{X}_e\}\} \\ &= \{\{\mathcal{X}_u, \mathcal{L}_v\}, \mathcal{X}_e\} + \{\mathcal{L}_v, \{\mathcal{X}_u, \mathcal{X}_e\}\} \\ &= -\{\mathcal{X}_{uv}, \mathcal{X}_e\} = 0. \end{aligned}$$

Proof that $\{\mathcal{X}_u, \mathcal{X}_e\} = 0$, i.e.,

$$\left\{ \langle x | \{\pi v \pi\} \rangle + \frac{n\mu^2}{(\text{tr} x)^2} \text{tr}(xv) - \mu i \frac{\text{tr}(x[v, \pi])}{\text{tr} x}, \langle x | \pi^2 \rangle + \frac{n\mu^2}{\text{tr} x} \right\} = 0.$$

This identity has terms up to degree four in μ . Again there is no need to verify the degree zero terms in μ .

For terms linear in μ , we have to verify that

$$\begin{aligned} 0 &= 4\{\langle S_{v\pi}x | \overline{\pi}, \overline{\langle \pi x | \pi \rangle}\rangle \\ &\quad - \frac{\mu i}{\text{tr} x} \{\text{tr}([x, v]\pi), \overline{\langle \pi^2 | x \rangle}\} - \frac{2\mu i}{\text{tr} x} \{\text{tr}([x, v]\pi), \overline{\langle \pi x | \pi \rangle}\} \\ &\quad + 2\mu i \frac{\text{tr}([x, v]\pi)}{(\text{tr} x)^2} \{\text{tr} x, \overline{\langle \pi x | \pi \rangle}\} \} \end{aligned}$$

i.e.,

$$\begin{aligned} 0 &= -8\mu i \frac{\text{tr}(x[S_{v\pi}x, \pi x])}{(\text{tr} x)^3} + \frac{n\mu i}{\text{tr} x} \langle \pi^2 | [x, v] \rangle \\ &\quad - \frac{2\mu i}{\text{tr} x} \text{tr}([\pi x, v]\pi) + 2\mu i \frac{\text{tr}([x, v]\pi)}{(\text{tr} x)^2} \text{tr}(\pi x) \end{aligned}$$

or

$$\begin{aligned} 0 &= -4\mu i \frac{\text{tr}(x[S_{v\pi}x, \pi])}{(\text{tr} x)^2} + \frac{n\mu i}{\text{tr} x} \langle \pi^2 | [x, v] \rangle \\ &\quad - \frac{2\mu i}{\text{tr} x} \text{tr}([\pi x, v]\pi) + 2\mu i \frac{\text{tr}([x, v]\pi)}{(\text{tr} x)^2} \text{tr}(\pi x) \end{aligned}$$

or

$$-2\text{tr}(x[S_{v\pi}x, \pi]) + \frac{1}{2}\text{tr } x \text{tr}(\pi^2[x, v]) - \text{tr } x\text{tr}([\pi x, v]\pi) + \text{tr}([x, v]\pi)\text{tr}(\pi x) = 0$$

or

$$-2\text{tr}(x[S_{x\pi}v, \pi]) + \frac{1}{2}\text{tr } x \text{tr}([\pi^2, x]v) - \text{tr } x\text{tr}([\pi x, v]\pi) + \text{tr}([x, v]\pi)\text{tr}(\pi x) = 0$$

or

$$-2S_{\pi x}([\pi, x]) + \frac{1}{2}\text{tr } x[\pi^2, x] - \text{tr } x[\pi, \pi x] + \text{tr}(\pi x)[\pi, x] = 0.$$

Since $2\pi x = \text{tr } \pi x + \text{tr } \pi x$, we have

$$-2S_{\pi x}([\pi, x]) + \frac{1}{2}\text{tr } x[\pi^2, x] + \frac{1}{2}\text{tr } \pi \text{tr } x[\pi, x] = 0.$$

So we need to verify that

$$S_{\pi x}([\pi, x]) = \frac{1}{4}\text{tr } x[\pi^2, x] + \frac{1}{4}\text{tr } \pi \text{tr } x[\pi, x]$$

which can indeed be verified:

$$\begin{aligned} S_{\pi x}([\pi, x]) &= \pi(x[\pi, x]) - x(\pi[\pi, x]) + (\pi x)[\pi, x] \\ &= \frac{1}{2}\pi[\pi, x^2] - \frac{1}{2}x[\pi^2, x] + (\pi x)[\pi, x] \\ &= \frac{1}{4}[\pi^2, x^2] - \frac{1}{4}[\pi^2, x^2] + (\pi x)[\pi, x] \\ &= (\pi x)[\pi, x] \\ &= \frac{1}{2}(\text{tr } x \pi + \text{tr } \pi x)[\pi, x] \quad \because \pi \text{ is a tangent vector} \\ &= \frac{1}{2}\text{tr } x \pi[\pi, x] + \frac{1}{2}\text{tr } \pi x[\pi, x] \\ &= \frac{1}{4}\text{tr } x[\pi^2, x] + \frac{1}{4}\text{tr } \pi[\pi, x^2] \\ &= \frac{1}{4}\text{tr } x[\pi^2, x] + \frac{1}{4}\text{tr } \pi \text{tr } x[\pi, x]. \end{aligned}$$

For terms quadratic in μ , we have to verify that

$$\begin{aligned} 0 &= 2\{\langle S_{v\pi}x|\pi\rangle, -\text{tr } x\} \frac{n\mu^2}{(\text{tr } x)^2} \\ &+ 2\left\{\frac{n\mu^2}{(\text{tr } x)^2}\text{tr}(xv), \langle \pi x|\pi\rangle\right\} - 4\frac{n\mu^2}{(\text{tr } x)^3}\text{tr}(xv)\left\{\text{tr } x, \langle \pi x|\pi\rangle\right\} \\ &- \frac{2\mu^2}{\text{tr } x}\{\text{tr}([x, v]\pi), \langle \pi x|\pi\rangle\}, \end{aligned}$$

i.e.,

$$\begin{aligned} 0 &= 2\text{tr}(S_{v\pi}x) \frac{n\mu^2}{(\text{tr } x)^2} \\ &+ 2\frac{n\mu^2}{(\text{tr } x)^2}\text{tr}((\pi x)v) - 4\frac{n\mu^2}{(\text{tr } x)^3}\text{tr}(xv)\text{tr}(\pi x) \\ &- \frac{4n\mu^2}{\text{tr } x} \frac{\text{tr}(x[x, v], \pi x)}{(\text{tr } x)^3} \end{aligned}$$

or

$$0 = 2\text{tr}(S_{v\pi}x) \frac{n\mu^2}{(\text{tr } x)^2}$$

$$+2 \frac{n\mu^2}{(\text{tr } x)^2} \text{tr}((\pi x)v) - 4 \frac{n\mu^2}{(\text{tr } x)^3} \text{tr}(xv) \text{tr}(\pi x) \\ - \frac{2n\mu^2}{\text{tr } x} \frac{\text{tr}(x[[x, v], \pi])}{(\text{tr } x)^2}$$

or

$$\text{tr}(S_{v\pi}x) \text{tr } x + \text{tr } x \text{tr}((\pi x)v) - 2 \text{tr}(xv) \text{tr}(\pi x) - \text{tr}(x[[x, v], \pi]) = 0$$

or

$$\text{tr}((v\pi)x) \text{tr } x + \text{tr } x \text{tr}((\pi x)v) = 2 \text{tr}(xv) \text{tr}(\pi x) + \text{tr}(x[[x, v], \pi])$$

or

$$2 \text{tr } x \text{tr}((\pi x)v) = 2 \text{tr}(xv) \text{tr}(\pi x) + \text{tr}([x, [x, v]]\pi).$$

Since $\frac{1}{2}[x, [x, v]] = \text{tr } x(xv) - \text{tr}(xv)x$, the preceding identity becomes

$$2 \text{tr } x \text{tr}((\pi x)v) = 2 \text{tr}(xv) \text{tr}(\pi x) + 2 \text{tr } x \text{tr}((xv)\pi) - 2 \text{tr}(xv) \text{tr}(\pi x)$$

which is trivially true.

For terms cubic in μ , we have to verify that

$$0 = \left\{ -\mu i \frac{\text{tr}(x[v, \pi])}{\text{tr } x}, \frac{n\mu^2}{\text{tr } x} \right\} \quad \text{or} \quad \{\text{tr}([x, v]\pi), \text{tr } x\} = 0$$

or

$$0 = \text{tr}([x, v])$$

which is trivially true.

There are no terms higher than cubic.

6. QUADRATIC RELATIONS

The main purpose of this section is to show that, in the Poisson realization for the conformal algebra of $H_n(\mathbb{C})$ that we have proved in the preceding section, the generators of the conformal algebra

$$\mathcal{S}_{u,v}, \quad \mathcal{X}_z, \quad \mathcal{Y}_w$$

satisfy some quadratic relations. Moreover, these quadratic relation is the consequence of a single one which shall be called the primary quadratic relation. As a consequence, for the corresponding U(1) Kepler problem, we obtain a formula connecting the Hamiltonian to the angular momentum and the Laplace-Runge-Lenz vector. This formula generalizes the one given by Eq. (2.8) of Ref. [11].

Theorem 3. *Let e_α be an orthonormal basis for $H_n(\mathbb{C})$. In the following we hide the summation sign over α or β . For the Poisson realization given by Eq. (4.3), we have the following*

(i) *primary quadratic relation*

$$(6.1) \quad \frac{2}{n} \mathcal{L}_{e_\alpha}^2 - \mathcal{L}_e^2 - \mathcal{X}_e \mathcal{Y}_e = -\mu^2$$

and secondary quadratic relations

$$(ii) \quad \mathcal{X}_{e_\alpha} \mathcal{L}_{e_\alpha} = n \mathcal{X}_e \mathcal{L}_e, \quad \mathcal{Y}_{e_\alpha} \mathcal{L}_{e_\alpha} = n \mathcal{Y}_e \mathcal{L}_e,$$

- (iii) $\frac{4}{n}\mathcal{L}_{e_\alpha,u}\mathcal{L}_{e_\alpha} = -\mathcal{X}_u\mathcal{Y}_e + \mathcal{X}_e\mathcal{Y}_u,$
- (iv) $\mathcal{X}_{e_\alpha}^2 = n\mathcal{X}_e^2, \mathcal{Y}_{e_\alpha}^2 = n\mathcal{Y}_e^2,$
- (v) $\frac{2}{n}\mathcal{L}_{e_\alpha,u}\mathcal{X}_{e_\alpha} = -\mathcal{X}_u\mathcal{L}_e + \mathcal{L}_u\mathcal{X}_e, \frac{2}{n}\mathcal{L}_{e_\alpha,u}\mathcal{Y}_{e_\alpha} = \mathcal{Y}_u\mathcal{L}_e - \mathcal{L}_u\mathcal{Y}_e,$
- (vi) $\mathcal{X}_{e_\alpha}\mathcal{Y}_{e_\alpha} = n(\mathcal{L}_e^2 + \mu^2),$
- (vii) $\frac{4}{n^3}\mathcal{L}_{e_\alpha,e_\beta}^2 = \mathcal{X}_e\mathcal{Y}_e - \mathcal{L}_e^2 + \frac{n-2}{n}\mu^2.$

Proof. (i) Since $\mathcal{L}_u = \langle ux|\pi\rangle$, we have

$$\begin{aligned}
\frac{2}{n}\mathcal{L}_{e_\alpha}^2 &= \frac{2}{n}\langle e_\alpha x|\pi\rangle^2 = \frac{2}{n}\langle e_\alpha|x\pi\rangle^2 = \frac{2}{n}\langle \pi x|\pi x\rangle \\
&= \frac{1}{2n}\|\text{tr } x\pi + \text{tr } \pi x\|^2 \\
&= \frac{1}{2n}((\text{tr } x)^2\|\pi\|^2 + (\text{tr } \pi)^2\|x\|^2 + 2\text{tr } x\text{tr } \pi\langle x|\pi\rangle) \\
&= \frac{1}{2n^2}((\text{tr } x)^2\text{tr } \pi^2 + (\text{tr } \pi)^2\text{tr } x^2 + 2\text{tr } x\text{tr } \pi\text{tr } (\pi x)) \\
&= \frac{(\text{tr } x)^2}{2n^2}(\text{tr } \pi^2 + 3(\text{tr } \pi)^2).
\end{aligned}$$

Since $\mathcal{L}_e = \langle x|\pi\rangle = \frac{1}{n}\text{tr } \pi\text{tr } x$,

$$\begin{aligned}
\mathcal{X}_e &= \langle x|\pi^2\rangle + \frac{n\mu^2}{\text{tr } x} = \langle \pi x|\pi\rangle + \frac{n\mu^2}{\text{tr } x} \\
&= \frac{1}{2}(\text{tr } \pi x + \text{tr } x\pi|\pi\rangle) + \frac{n\mu^2}{\text{tr } x} \\
&= \frac{\text{tr } x}{2n}(\text{tr } \pi^2 + (\text{tr } \pi)^2) + \frac{n\mu^2}{\text{tr } x}
\end{aligned}$$

and $\mathcal{Y}_e = \langle e|x\rangle = \frac{1}{n}\text{tr } x$, so

$$\mathcal{L}_e^2 + \mathcal{X}_e\mathcal{Y}_e = \frac{(\text{tr } x)^2}{2n^2}(\text{tr } \pi^2 + 3(\text{tr } \pi)^2) + \mu^2.$$

The primary quadratic relation is then clear.

- (ii) The two identities can be obtained by taking the Poisson bracket of the primary quadratic relation with \mathcal{X}_e and \mathcal{Y}_e respectively. For example, since

$$\frac{2}{n}\sum\{\mathcal{L}_{e_\alpha}^2, \mathcal{X}_e\} - \{\mathcal{L}_e^2, \mathcal{X}_e\} - \{\mathcal{X}_e\mathcal{Y}_e, \mathcal{X}_e\} = 0,$$

we have

$$\frac{4}{n}\mathcal{L}_{e_\alpha}\mathcal{X}_{e_\alpha} - 2\mathcal{L}_e\mathcal{X}_e - \mathcal{X}_e \cdot 2\mathcal{L}_e = 0$$

or $\mathcal{L}_{e_\alpha}\mathcal{X}_{e_\alpha} = n\mathcal{X}_e\mathcal{L}_e$.

- (iii) The identity can be obtained by taking the Poisson bracket of the primary quadratic relation with \mathcal{L}_u .
- (iv) By taking the Poisson bracket of $\mathcal{L}_{e_\alpha}\mathcal{X}_{e_\alpha} = n\mathcal{X}_e\mathcal{L}_e$ with \mathcal{X}_e , we have $\mathcal{X}_{e_\alpha}^2 = n\mathcal{X}_e^2$. Similarly, we can prove $\mathcal{Y}_{e_\alpha}^2 = n\mathcal{Y}_e^2$.
- (v) The two identities can be obtained by taking the Poisson bracket of the identity in (iii) with \mathcal{X}_e and \mathcal{Y}_e respectively.
- (vi) The identity can be obtained by taking the Poisson bracket of the first identity in (ii) with \mathcal{Y}_e and then using the primary quadratic relation.

(vii) Taking the Poisson bracket of the first identity in (v) with \mathcal{Y}_u and then taking $u = e_\beta$ and summing over β , we get

$$\frac{2}{n}(\mathcal{Y}_{L_{e_\alpha, e_\beta}(e_\beta)}\mathcal{X}_{e_\alpha} - 2\mathcal{L}_{e_\alpha, e_\beta}^2) = 2\mathcal{L}_{e_\beta}^2\mathcal{L}_e + \mathcal{X}_{e_\beta}\mathcal{Y}_{e_\beta} - \mathcal{Y}_{e_\beta}^2\mathcal{X}_e - 2\mathcal{L}_{e_\beta}^2$$

or

$$\frac{2}{n}(\mathcal{Y}_{e_\alpha e_\beta^2 - L_{e_\beta}^2 e_\alpha}\mathcal{X}_{e_\alpha} - 2\mathcal{L}_{e_\alpha, e_\beta}^2) = 2n^2\mathcal{L}_e^2 + \mathcal{X}_{e_\beta}\mathcal{Y}_{e_\beta} - n^2\mathcal{Y}_e\mathcal{X}_e - 2\mathcal{L}_{e_\beta}^2$$

Since $e_\alpha^2 = n^2e$ and $L_{e_\beta}^2 = \frac{n^2}{2}(L_e + |e\rangle\langle e|)$ (see line 13 after equation (6.23) of Ref. [13]), we have $e_\alpha e_\beta^2 - L_{e_\beta}^2 e_\alpha = \frac{n^2}{2}(e_\alpha - \langle e|e_\alpha\rangle e)$, so, in the preceding equation,

$$LHS = n(\mathcal{Y}_{e_\alpha}\mathcal{X}_{e_\alpha} - \mathcal{X}_e\mathcal{Y}_e) - \frac{4}{n}\mathcal{L}_{e_\alpha, e_\beta}^2.$$

Therefore, we have

$$-\frac{4}{n}\mathcal{L}_{e_\alpha, e_\beta}^2 = 2n^2\mathcal{L}_e^2 + (1-n)\mathcal{X}_{e_\alpha}\mathcal{Y}_{e_\alpha} + n(1-n)\mathcal{Y}_e\mathcal{X}_e - 2\mathcal{L}_{e_\alpha}^2.$$

Upon using identities in (i) and (vi), the preceding identity becomes

$$-\frac{4}{n}\mathcal{L}_{e_\alpha, e_\beta}^2 = 2n^2\mathcal{L}_e^2 + (1-n)n(\mathcal{L}_e^2 + \mu^2) + n(1-n)\mathcal{Y}_e\mathcal{X}_e - n(\mathcal{L}_e^2 + \mathcal{X}_e\mathcal{Y}_e - \mu^2)$$

or

$$\frac{4}{n^3}\mathcal{L}_{e_\alpha, e_\beta}^2 = -\mathcal{L}_e^2 + \mathcal{X}_e\mathcal{Y}_e + \frac{n-2}{n}\mu^2.$$

□

Remark 6.1. In Ref. [13], the quantum version of these quadratic relations is worked out for un-magnetized generalized Kepler problems associated with arbitrary simple Euclidean Jordan algebra. As far as we know, the quantum version of these quadratic relations for the MICZ-Kepler problems appeared first in Ref. [14]. The fact that these quadratic relations are consequences of a single quadratic relation was observed first in Ref. [13]. Please also compare with the relevant part in Refs. [15, 16, 17, 5, 18].

As a corollary of these quadratic relations, let us derive a formula connecting the Hamiltonian to the angular momentum and the Laplace-Runge-Lenz vector. From Ref. [12] we know that the Hamiltonian is

$$H = \frac{1}{2}\frac{\mathcal{X}_e}{\mathcal{Y}_e} - \frac{1}{\mathcal{Y}_e},$$

the angular momentum is $\mathcal{L}_{e_\alpha, e_\beta}$, and Laplace-Runge-Lenz vector is

$$\mathcal{A}_{e_\alpha} = \frac{1}{2}\left(\mathcal{X}_{e_\alpha} - \mathcal{Y}_{e_\alpha}\frac{\mathcal{X}_e}{\mathcal{Y}_e}\right) + \frac{\mathcal{Y}_{e_\alpha}}{\mathcal{Y}_e}.$$

Corollary 1. Let e_α be an orthonormal basis for $H_n(\mathbb{C})$, $L^2 = \frac{1}{2}\sum_{\alpha, \beta}\mathcal{L}_{e_\alpha, e_\beta}^2$, and $A^2 = -1 + \sum_{\alpha}\mathcal{A}_{e_\alpha}^2$. Then the Hamiltonian H satisfies the relation

$$(6.2) \quad -2H\left(L^2 - \frac{n^2(n-1)}{4}\mu^2\right) = \left(\frac{n}{2}\right)^2(n-1-A^2).$$

Proof. For simplicity, we shall hide the summation sign Σ in the proof below. Since $\mathcal{A}_{e_\alpha} = \frac{1}{2}\left(\mathcal{X}_{e_\alpha} - \mathcal{Y}_{e_\alpha}\frac{\mathcal{X}_e}{\mathcal{Y}_e}\right) + \frac{\mathcal{Y}_{e_\alpha}}{\mathcal{Y}_e}$, using quadratic relations in the previous theorem, we have

$$\mathcal{A}_{e_\alpha}^2 = n + \frac{1}{4}\mathcal{X}_{e_\alpha}^2 - \frac{1}{2}\mathcal{X}_{e_\alpha}\mathcal{Y}_{e_\alpha}\frac{\mathcal{X}_e}{\mathcal{Y}_e} + \frac{n}{4}\mathcal{X}_e^2 + \mathcal{X}_{e_\alpha}\frac{\mathcal{Y}_{e_\alpha}}{\mathcal{Y}_e} - n\mathcal{X}_e \quad \text{using } \mathcal{Y}_{e_\alpha}^2 = n\mathcal{Y}_e$$

$$\begin{aligned}
&= n + \frac{n}{2}\mathcal{X}_e^2 + \mathcal{X}_{e_\alpha}\mathcal{Y}_{e_\alpha}\left(-\frac{\mathcal{X}_e}{2\mathcal{Y}_e} + \frac{1}{\mathcal{Y}_e}\right) - n\mathcal{X}_e \quad \text{using } \mathcal{X}_{e_\alpha}^2 = n\mathcal{X}_e^2 \\
&= n + (n\mathcal{Y}_e\mathcal{X}_e - \mathcal{X}_{e_\alpha}\mathcal{Y}_{e_\alpha})H \quad \text{using } H = \frac{\mathcal{X}_e}{2\mathcal{Y}_e} - \frac{1}{\mathcal{Y}_e} \\
&= n + n(\mathcal{Y}_e\mathcal{X}_e - \mathcal{L}_e^2 - \mu^2)H \quad \text{using identity (vi) in Theorem 3} \\
&= n + n\left(\frac{8}{n^3}L^2 - \frac{2(n-1)}{n}\mu^2\right)H \quad \text{using identity (vii) in Theorem 3} \\
&= n + \frac{8}{n^2}H\left(L^2 - \frac{n^2(n-1)}{4}\mu^2\right)
\end{aligned}$$

Since $A^2 = -1 + \sum \mathcal{A}_{e_\alpha}^2$, we are done. \square

Remark 6.2. If $n = 2$, Eq. (6.2) becomes

$$-2H(L^2 - \mu^2) = (1 - A^2),$$

i.e., the formula given by Eq. (2.8) of Ref. [11]. Please also compare with Eq. (6.10) in Ref. [5].

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DIVISION OF SCIENCE AND MATHEMATICS, NEW YORK UNIVERSITY ABU DHABI, PO BOX 129188,
ABU DHABI, UNITED ARAB EMIRATES.

E-mail address: sofiane.bouarroudj@nyu.edu

DEPARTMENT OF MATHEMATICS, HONG KONG UNIV. OF SCI. AND TECH., CLEAR WATER BAY,
KOWLOON, HONG KONG.

E-mail address: mameng@ust.hk